A Book Stacking Problem

For Calculus II

By Dr. W. L. Culbertson

If you stack up books, how far over the edge we can push things? Stacking up books in straight stacks is for librarians. Just like that tower in Pisa, mathematicians like to make their stacks lean and see just how much they can get away with making stacks bend beyond the vertical. How far out over the edge can we get the top book of leaning stack?

First of all, if we start with one book at the edge of the table, we can push it just half way over the edge until it topples to the floor. Okay, let's agree to stop just the tiniest fraction of an inch before that happens. With one book of mass *m* and length *b*, the balance point is at the edge of the table. Let's call that $x = 0$, and edge of the book is at a length, $\frac{1}{2}b$, beyond it.

Lay a second book on the table with its right side at the edge of the table and place the top book with its center balance point at the edge of the table, $x = 0$. What is the balance point of this two-book system?

Review of finding the balance point:
The moment of turning force, *M*, about a balance point at $x = 0$ is the product of the mass of the object, *m*, times its distance from the zero point, *x*. In equation form: $M = mx$. If there are a series of *n* masses, the total moment or the total turning force, M_n, is just the sum of the *n* individual moments:

$$M_n = m_1x_1 + m_2x_2 + m_3x_3 + \cdots + m_nx_n \quad \text{or} \quad M_n = \sum_{i=1}^{n} m_ix_i$$

The center of mass of the system, $\bar{x}$, is the balance point — where the moments on the left side equal the moments on the right side. Each of the x_i points is at a distance $(x_i - \bar{x})$ from the balance point — x_i's to the left of the balance point will have negative differences, and those to the right will be positive. Therefore, the sum of the moments around that balance point must be zero. In other words:

$$\sum_{i=1}^{n} m_i(x_i - \bar{x}) = 0$$

To find where the balance point of a system of *n* masses is located, solve this equation for $\bar{x}$:

$\sum_{i=1}^{n} m_i(x_i - \bar{x}) = 0$	The system balanced around point $\bar{x}$.
$\sum_{i=1}^{n} (m_ix_i - m_i\bar{x}) = 0$	Distribute the multiplication.

$$\sum_{i=1}^{n} m_i x_i - \sum_{i=1}^{n} m_i \overline{x} = 0$$ Using the properties of summations.

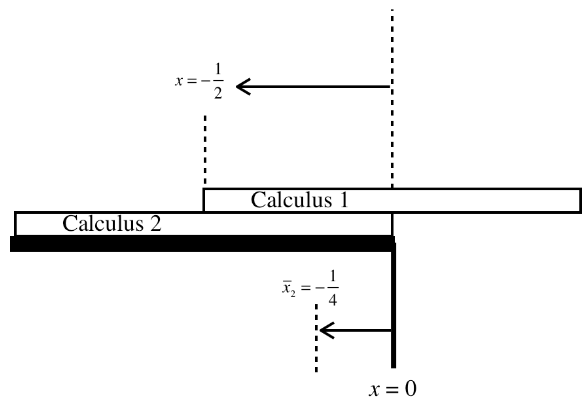
$$\sum_{i=1}^{n} m_i x_i - \overline{x} \cdot \sum_{i=1}^{n} m_i = 0$$ Factor the constant $\overline{x}$ over the summation.

$$\overline{x} = \frac{\sum m_i x_i}{\sum m_i} = \frac{M_n}{\sum m_i}$$ Solving for $\overline{x}$

This means the center of balance of the system is the total moments divided by the total mass of the system.

Back to the two-book system:

Now we have two books — one book on the table and the other on top pushed right to its balance point. The new book (book 2) is placed under book 1 with its right edge at the right edge of the table. The new book's center of mass is at $-\frac{1}{2}b$ (recall the edge of the table is $x = 0$). The top book has its center of mass right at the zero point, so the center of mass of the two-book system would be the total moments divided by the total mass:



$\overline{x} = \left[m_2 \cdot \left(-\frac{1}{2}b\right) + m_1 \cdot 0\right] \div \left[m_2 + m_1\right]$

But, let's keep it simple. Let's assume we are stacking up our math books. That way all the masses are the same, and who would really care if we goof and the whole stack topples over? The balance point is then:

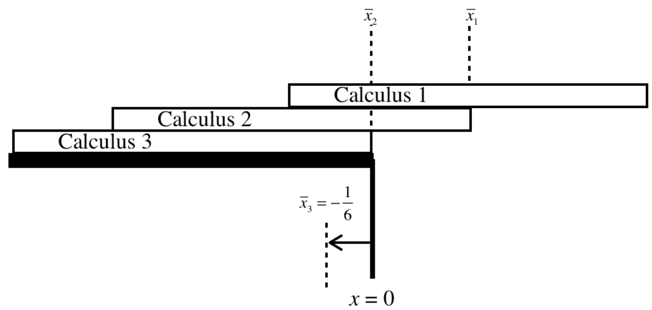
$$\overline{x}_2 = \frac{-\frac{1}{2}bm}{2m} = -\frac{1}{4}b$$

In fact, let's make things even simpler by doing everything in units of book lengths — we'll let $b = 1$. Now we can see that the stack of two books has a center of balance at $\overline{x}_2 = -\frac{1}{4}$. That means we can move the stack a quarter of a book-length to the right before it starts to fall off the table.

Now where is the center of the top book (let's call it c)? We started with its center over $x = 0$ so its center is now a quarter of a book-length beyond that at $c_2 = +\frac{1}{4}$. That means that three-fourths of the top book is hanging beyond the edge of the table.

Three books:

Lay a third book on the table with its right edge at the edge of the table. Take the two-book system and set it on top with its balance point at the edge of the table. Where is the balance point of the three-book system?



Since we laid the two-book stack with its balance point at the third book's right-hand edge, we can treat the two-book stack as a single, two-mass weight right at $x = 0$. The center of mass for the three-book system is

$$\bar{x}_3 = \left[m \cdot \left(-\tfrac{1}{2}\right) + 2m \cdot 0\right] \div \left[m + 2m\right] = -\frac{\frac{1}{2}}{3} = -\frac{1}{6}$$

Since the new balance point is left of the edge of the table, we can move the whole stack $\frac{1}{6}$ of a book length farther to the right. The top book? It's center is now $c_3 = \frac{1}{4} + \frac{1}{6}$.

Four books:

By a similar argument, $\bar{x}_4 = \left[m \cdot \left(-\tfrac{1}{2}\right) + 3m \cdot 0\right] \div \left[m + 3m\right] = -\frac{\frac{1}{2}}{4} = -\frac{1}{8}$ and $c_4 = \frac{1}{4} + \frac{1}{6} + \frac{1}{8}$.

Now the pattern:

If we could continue this process, we would see the center of the top book, *c*, creeping out with the series:

$$c_n = \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \cdots \frac{1}{2(n+1)}$$

As is our wont, let's continue the series through an infinite number of math books (There does not appear to be any limit on the number of them.) and see where the center ends up.

$$c_\infty = \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \cdots = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\right)$$

Or, in more compact notation,

$$c_\infty = \frac{1}{2}\sum_{n=2}^{\infty}\frac{1}{n} \quad \text{or} \quad c_\infty = \frac{1}{2}\left[\left(\sum_{n=1}^{\infty}\frac{1}{n}\right) - 1\right]$$

Notice that series? It's the harmonic series, and we all know the harmonic series diverges — its sum is infinity. Okay, we have the series multiplied by $\frac{1}{2}$, but half of infinity is . . . In other words, we can get that top book out past the edge of the table as far as we want it to be. All we need is enough books.

How many books would it take . . .
So how many books would it take before we could get the top book balanced at a point where it is completely beyond the edge of the table? We would have to have the center of the top book moved to $x = \frac{1}{2}$. We derived the expression for the center of the top book as:

$$c_n = \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \cdots \frac{1}{2(n+1)}$$

With a little arithmetic we see

$$c_4 = \frac{1}{4} + \frac{1}{6} + \frac{1}{8} = \frac{13}{24} = .541666\cdots$$

Using four books, we could build a stack with the top book completely over the edge. One word about stacking books, however. Don't try these tricks at home, kids. Even though books are fairly stiff, paper and cardboard are somewhat compressible. As more and more books are stacked up with their weight concentrated towards the edges of the ones below, the stack will have a tendency to lean a little outwards. A taller stack will have the most compression effect at the bottom where the pressure is greatest. A small lean at the bottom gets magnified by the length of the lever arm of the stack producing a larger shift at the top — just where it will cause the most problems trying to balance that last book. Maybe we should just imagine using books with the properties of incompressible steel plates?

When would the stack be able to support a top book with its center one whole book-length (about 11 inches) beyond the table top? The arithmetic gets a bit onerous, but it turns out 11 books would do the trick.

What about two book lengths? When you start on this little project, you realize that although the harmonic series diverges, it diverges increasingly slowly as it goes. Computing those repetitive fraction sums is tiresome (to say the least).

It turns out that the rate of divergence is at a logarithmic pace. Leonard Euler showed this in 1735 and found that the harmonic series, H_n

$$H_n = \sum_{n=1}^{k} \frac{1}{n} = \ln k + \gamma + \varepsilon_k$$

Nothing comes without some sort of price, and here we have two new terms.
The term $\varepsilon_k \sim \frac{1}{2k}$ and $\lim_{k\to\infty} \varepsilon_k = 0$.

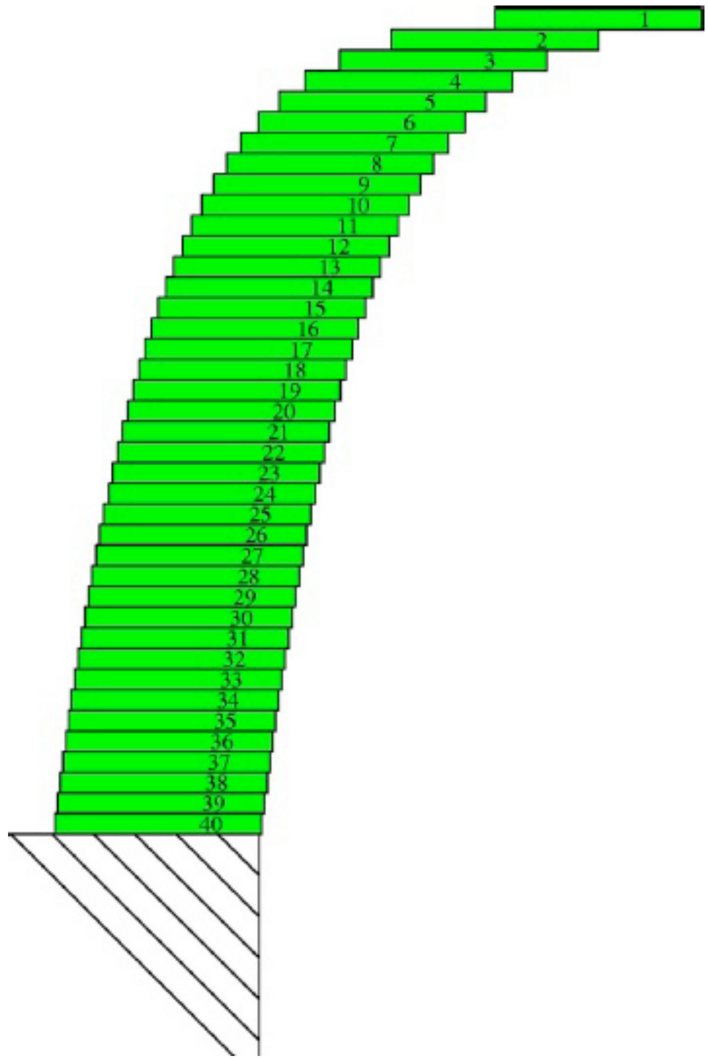
The other term, $\gamma = .57721\,56649\,01532\cdots$, is the Euler-Mascheroni Constant. This number shows up in a large number of equations in higher mathematics. Interestingly, this number is not known to be algebraic (a solution to a polynomial) or transcendental (non-algebraic irrational numbers like π, *e*, and ϕ). In fact, mathematicians are not even sure it is irrational. If it is a rational number, the integer denominator must be greater than $10^{242,080}$. Because it shows up in so many places, this question of the irrationality of γ is a major open question in mathematics.

Now using our result $c_\infty = \frac{1}{2}\left[\left(\sum_{n=1}^{\infty}\frac{1}{n}\right)-1\right]$, we could write the location of the center of the top book c_k as

$$c_k = \frac{1}{2}\left[H_k - 1\right] \sim \ln k + \frac{1}{2k} - .4227843351 \quad \text{(after just 6 terms the error < .01\%)}$$

Solving the approximation for *k*, we find it would take 83 books before the top book would project out over the edge of the table more than two book-lengths. Let's see. Our book is about $1\frac{1}{2}$ inches thick which means the stack would be over ten feet tall.

Here is a graphic of the logarithmic progression of the book stack.



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