Introduction to Logarithms

What Is a Logarithm?

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In the early 1600s, arithmetic was in a crisis. Now that scientists were convinced the Earth and other planets revolved around the sun, they could accurately compute the orbits of the planets. Columbus and the age of discovery had opened the world to business, and mariners sailed off to navigate newly developed trade routes around the world. Engineers applied expanding scientific understanding of material and structural forces to plan and build increasingly ambitious projects.

All this was good news, but scientists, navigators, and engineers faced long and tedious computational problems like:

$$p = \sqrt{\frac{(39.38)(279.4)(8.488)^3}{4.033}} \quad \text{(calculating an orbital period, say)}$$

This calculating task might not impress students today, but back in the 1600s these computations had to be done by hand! This was the B.C. era—Before Calculators. If you want some fun, try this computation without a calculator. You might have to look up the algorithm for extracting a square root—by hand. Work at this computation awhile, and you will begin to appreciate just what the words "long and tedious" really mean. Oh—don't forget to check your work!

This hand labor motivated mathematicians to develop an easier way to do long, hard computations. To this end, John Napier, a Scottish mathematician and 8th Laird of Merchistoun, introduced the idea of logarithms. Using logarithms, calculations could be simplified by using the laws of exponents. The laws of exponents? Let's review. When exponential numbers have the same base, say the number *a*:

$a^n \cdot a^m = a^{(n+m)}$	To <u>multiply</u> exponential numbers: **ADD** the exponent parts
$\dfrac{a^n}{a^m} = a^{(n-m)}$	To <u>divide</u>: **SUBTRACT** the exponent parts
$\left(a^n\right)^m = a^{n \cdot m}$	To take an exponential number to a power: **MULTIPLY** the exponents

So how did Napier use the laws of exponents? Consider the multiplication problem:

$$(32{,}768)(4096) = ?$$

We could do the problem by hand using the algorithm:

$$\begin{array}{r} 32,768 \\ \underline{\times 4096} \\ \text{Partial Product 1} \\ \text{Partial Product 2}\quad \\ \vdots\quad\quad\quad \\ \hline \text{Sum of Partial Products} \end{array}$$

which would be "fun."

Napier's idea worked like this. Instead of multiplying the numbers long-hand, work with the exponent parts. This example problem, $(32,768)(4096) = ?$, is a special construct. The numbers 32,768 and 4096 just happen to be powers of 2. This problem could be written:

$$(32,768)(4096) = 2^{15} \cdot 2^{12}$$

Therefore, we can do this particular problem by using the laws of exponents:

$$(32,768)(4096) = 2^{15} \cdot 2^{12} = 2^{(15+12)} = 2^{27} = 134,217,728$$

A long involved multiplication problem becomes a simple addition problem. The only hand computation needed was $15+12$. But wait—how do we know those exponent equivalents? The answer is, we can look them up. Somebody already figured out $2^{27} = 134,217,728$, and we can look up that fact anytime we need it. Of course this is a special problem, namely one that can be easily written in exponential form.

Here is Napier's had his big idea: We can write <u>any</u> number in exponential form! We can choose a base, then figure out what power to raise that base to in order to get the number we want. Take the number 5. Suppose we want to write 5 in exponential form using 10 as a base number. In other words:

$$5 = 10^{?}$$

What power do we need? Well, the power 0 is too small because $10^0 = 1$. The power 1 is too big because $10^1 = 10$. Therefore, the answer must be between 0 and 1. However, this is not a "Goldilocks" problem where the answer is half-way in between. No, we need the power .6990:

$$10^{.6990} = 5.00034535\ldots \approx 5$$

Calculator Check: Before we go any farther, it is time for a calculator check. How do we take 10 to the .6990 power? Different calculators use different systems, but most current calculators use some sort of algebraic entry system. You enter the problem just as you would write it. Scientific calculators have an exponent (a to-the-power-of) key. Many calculators use the key labeled [^], the ^ symbol meaning raised "up" to a power. You would enter the problem:

Base number [^] Power [=].

On your calculator, enter:

10 [^] .6990 [=] .

The calculator should display: 5.00034535 which is the answer above. If your calculator uses different symbols or entry system, check the manual or check with someone who knows your calculator.

So where did that exponent .6990 come from? This is a backwards power question like these:

$8 = 2^?$ Since we know $2 \cdot 2 \cdot 2 = 2^3 = 8$, the exponent is 3.
$16 = 2^?$ Since we know $2 \cdot 2 \cdot 2 \cdot 2 = 2^4 = 16$, the exponent is 4.
$16 = 4^?$ Since we know $4 \cdot 4 = 4^2 = 16$, the exponent is 2.
$27 = 3^?$ Since we know $3 \cdot 3 \cdot 3 = 3^3 = 27$, the exponent is 3.

We can use any base we want, but the trick is to find the power. When we represent a number in exponential form, the power number is called the **logarithm** of the original number. Of course the power number is only relevant if we know what base it is applied to. The above examples are logarithm problems. In the first example we had to figure out the power of 2 needed to get 8:

$8 = 2^?$ $\Rightarrow$ Same as saying: what is the logarithm of 8, base 2?

Mathematicians will never write something in words where they can invent a symbol to save writing time. Here is the mathematical notation:

$\log_2 8 = 3$ Read: "log, base 2, of 8 is **3**." Means: **3** is the power of 2 which makes 8.

The expression, $\log_2 8$, is read, log, base 2, of 8. It is an operation that finds the power of 2 that equals 8. Logarithms are backwards exponent questions. Here are the rest of the examples listed above in logarithmic notation:

$16 = 2^?$ $\Rightarrow$ $\log_2 16 = ?$ (Answer is 4)
$16 = 4^?$ $\Rightarrow$ $\log_4 16 = ?$ (Answer is 2)
$27 = 3^?$ $\Rightarrow$ $\log_3 27 = ?$ (Answer is 3)

And let's not forget:

$$5 = 10^{?} \quad \Rightarrow \quad \log_{10} 5 = ? \quad \text{(Answer is .6990)}$$

Now we can use logarithms to simplify the arithmetic computations needed for a problem like:

$$p = \sqrt{\frac{(39.38)(279.4)(8.488)^3}{4.033}}$$

Mathematicians using Napier's idea could change all the numbers to logarithms, base 10, then do calculations using the laws of exponents. That calculation would give the exponent of the answer, then they would change it back to standard, numerical form.

Did you notice how we conveniently ignored one important question? Just HOW did they find the logarithms of those numbers? The answer is, it's complicated—very complicated. However, once the work is done, the logarithm of a number becomes a known fact. You can look it up. In fact, that's just what they did. It took a lot of work, and a lot of people spent a lot of time calculating logarithms—like $\log_{10} 5 = .6990$. Once the values were computed, they collected all the results into large tables.

There are some patterns that make the tables a lot simpler; you do not really have to look at all possible numbers. You can just use the numbers between 1 and 10. We will not worry about the details here. (After all, we do not look up logs anymore since calculators do the work for us.) If you look at a couple of examples, you should get the idea:

$$\log_{10} 5 = .6990$$
$$\log_{10} 50 = 1.6990$$
$$\log_{10} 500 = 2.6990$$
$$\log_{10} 5000 = 3.6990$$

Once these "log tables" were compiled, they made large computations like the one above much quicker and easier. In our problem from the 1600s, there was a multiplication like this:

$$(39.38)(279.4) = 11{,}002.772$$

We could multiply this out by hand. (There would be four partial products to add. Count the total decimal places in the problem and put that many places in the answer.) Using logarithms, we change the two numbers to exponential form (by looking in the tables) then compute exponent of the answer by adding the exponents. Once added, we change the exponent part of the answer back to the answer number itself (by looking in the tables).

$$(39.38)(279.4) \overset{\text{Tables}\rightarrow}{=} \left(10^{1.5953}\right)\left(10^{2.4462}\right) = 10^{(1.5953+2.4462)} = 10^{4.0415} \overset{\text{Tables}\rightarrow}{=} 11{,}002.718$$

Changing the numbers into and out of powers of 10 can be done by looking them up in a table. Adding the exponents 1.5953 to 2.4462—*by hand*—is considerably easier and quicker than multiplying the two four-digit numbers together—*by hand*. However, notice that there is a trade off. The answer we get using the exponential forms is not *exactly* the correct answer. In the tables, the exact exponent was rounded to four decimal places, so there is some error. More accurate tables make for more accurate answers. How much accuracy do we need? Trajectories for the Apollo moon flights were calculated with about this degree of accuracy, and that seemed to work.

What about a computation with a power? The computation above has a cubic power to calculate:

$$(8.488)^3$$

Instead of multiplying (remember, we are computing by hand here!):

$$(8.488)^3 = (8.488)(8.488)(8.488) = 611.527670272$$

we can use logarithms to write the problem in exponential form:

$$(8.488)^3 \overset{\text{Tables}\rightarrow}{=} \left(10^{0.9288}\right)^3$$

In exponential form, we do the problem by multiplying the exponents:

$$(8.488)^3 \overset{\text{Tables}\rightarrow}{=} \left(10^{0.9288}\right)^3 = 10^{(.9288\times 3)} = 10^{2.7864} \overset{\text{Tables}\rightarrow}{=} 611.5050$$

Changing the numbers into and out of powers of 10 can be done by looking numbers up in a table. In exponential form, the computation is multiplying 0.9288 by 3—*by hand*—considerably easier, not to mention quicker, than multiplying three factors of 8.488 together—*by hand*. Again, a rounding error shows up in the answer, but we are off by only about 0.04%. Not too bad for the amount of work we save.

So,with paper and pencil, either calculate by hand:

$$p = \sqrt{\frac{(39.38)(279.4)(8.488)^3}{4.033}}\ ,$$

or look up the logarithms of the four numbers, and calculate by hand the exponent part of the answer:

$$\tfrac{1}{2}(1.5953 + 2.4462 + 3\times .9288 - .6056) \quad = \quad ?$$

Finally, look up that exponent of the answer in the log table—going the opposite direction—and there is your answer. Okay, there is still some arithmetic to do, but it is **WAY, WAY EASIER** arithmetic.

We do not use logarithms to do calculations now that we have electric brains, our calculators, to do the hard work. The key thing to remember is that when we find the logarithm of a number, that logarithm is an exponent. The operation of finding a logarithm is a mathematical operation which has special properties that make it a useful mathematical tool.

Logarithms are not just an antiquated computational tool made obsolete by electronic calculators. No, logarithms are quite useful in their own right for describing a variety of mathematical relationships.

One area where logarithms are useful is for describing the sensitivity of our senses. Our senses of touch, sight, hearing, and so on seem to respond on what is called a **logarithmic scale**. The range of sensory levels we can detect with our senses is huge: a breath of a whisper to the body shaking roar of a rock concert, the twinkling light of a star in the sky on a summer's night to the glare of sunlight on new-fallen snow.

If we measured the sound energy present in a quiet library and assign that an energy level $=1$ unit, then an ordinary conversation has sound energy $=100$, a vacuum cleaner $=1000$, and a chain saw $=10{,}000{,}000$ units. The numbers get so big so fast, it is difficult to compare them. We compare the loudness of sound the way our ears respond—logarithmically. The decibel scale is a logarithmic scale where the library $=40\ \text{dB}$, conversation $=60\ \text{dB}$, a vacuum cleaner $=70\ \text{dB}$, and a chain saw $=110\ \text{dB}$. By comparing the exponents of the measures on a logarithmic scale, there is enough difference in the units at the low end to compare a pin drop to a whisper while at the same time keeping the measure of the sound level at a rock concert from being a number too large to be easily understandable.

Astronomers measure brightness of objects in the night sky on the magnitude scale. This scale compares stars on the basis of the logarithm of the energy received. Why logarithms? It is the way our eyes work. Our perception of a star's brightness is based on the logarithm of the energy from the star.

In chemistry, the strength of an acid or base is related to the number of hydrogen ions available for action. Just as with sound energy, the numbers of ions changes dramatically. Rather than deal with an extremely large numbers, pH is measured on a logarithmic scale.

Logarithms and logarithmic functions are useful mathematically, and they are important in many areas of science and engineering. But keep in mind, logarithms are involved in the way you see the world—literally.