Completing the Square

W. L. Culbertson, PhD

Completing the square is a technique for changing the form of a quadratic equation or expression so the variable is expressed as the square of a binomial plus (minus) some constant term. Here is a quadratic in binomial square form:

$$(x+3)^2-4$$

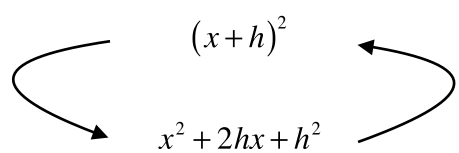
If we square out the binomial, we would get:

$$(x+3)(x+3)-4$$
$$x^2+3x+3x+9-4$$
$$x^2+6x+5$$

When we complete the square, we want to move <u>backwards</u> from x^2+6x+5 to get the binomial square form. How do we do this? First, we must remember the pattern we get when we square a binomial:

$(x+h)^2$	
$(x+h)(x+h)$	Multiply the binomial by itself
$x^2+hx+hx+h^2$	FOIL product
$x^2+2hx+h^2$	Collect like terms

But this is a two-way process. If we square a binomial $(x+h)^2$ we get the trinomial $x^2+2hx+h^2$. Therefore, if we have the trinomial $x^2+2hx+h^2$ it must factor into $(x+h)^2$



If we start with the trinomial (above) x^2+6x+5 we need to manipulate part of it into the form $x^2+2hx+h^2$ so we can factor that part into a binomial.

Complete the square:

x^2+6x+5		Starting trinomial
x^2+6x	$+5$	Add space to complete the square
$x^2+2hx+h^2$		The pattern we need to match

Note the x^2 term matches in both the given trinomial and the pattern we need. The key is the coefficient on the x terms. The coefficient of the given trinomial is 6 (as seen in the $6x$ term). The pattern coefficient is $6h$ (as seen in the $6hx$ term). Since the coefficients of the x terms must match, to complete the square we need to know the value of h. We can figure that out since:

$6x = 2hx$	Required to match pattern
$6 = 2h$	Coefficient parts must match
$h = 3$	Solve to find the value of h needed

The h number of the pattern is 3, but to have a complete trinomial which we can factor, we must complete the pattern: $x^2 + 2hx + h^2$.

$x^2 + 6x \ + \ ? \qquad +5$	Ready to complete square pattern
$x^2 + 2hx + h^2$	The pattern we need to match
$h^2 = (3)^2 = 9$	Since $h = 3$, the h^2 number we need is 9

How do we complete the pattern? It depends on the form of the trinomial we are given. In this example we only have a trinomial **expression** — no equation. We'll see how to complete a square in an equation next. With the expression we have, we need a +9 to complete the square. Since this is **not** an equation, we cannot just add a 9 — that would change the value. Instead, we will add 'zero' which will not change the value of the expression. However, we will chose to write zero in a special way to allow us to complete the square.

$0 = +9 - 9$	Hopefully this is not surprising.
$x^2 + 2hx + h^2$	The pattern we need to match
$x^2 + 6x \ +9 - 9 + 5$	Add 'zero' to complete the square
$(x^2 + 6x \ +9) - 9 + 5$	Select the three terms which form the perfect square
$(x+3)^2 - 9 + 5$	Factor the trinomial square
$(x+3)^2 - 4$	Collect number terms to finish

Note we can't solve anything here because our expression is not 'equal to' anything. We're done.

Solving by Completing the Square

Suppose we have the equation (notice the = sign?), $x^2 + 6x + 5 = 0$, and want to complete the square to solve for x. We match the pattern to find the h number as above. Since we have an equation, we can complete the square pattern by adding h^2 to both sides of the equation.

$x^2 + 6x + 5 = 0$	Equation in which we want to complete the square
$x^2 + 6x \quad = -5$	Isolate the x terms by moving 5 to the other side
$x^2 + 6x + 9 = -5 + 9$	Add 9 to both sides
$x^2 + 6x + 9 = 4$	We now have a trinomial on the left which factors
$(x+3)^2 = 4$	Factor to a binomial square on the left side
$(x+3)^2 = 4$	To solve for x we can take the square root to get at the x by 'unsquaring'
$\sqrt{(x+3)^2} = \pm\sqrt{4}$	Remember, both positive and negative values!
$x + 3 = \pm 2$	Simplify
$x = -3 \pm 2$	Solve for x
$x = -1, x = -5$	The solution has two values